

Reducing system of parameters and the Cohen–Macaulay property

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Abstract. Let R be a local ring and let (x_1, \dots, x_r) be part of a system of parameters of a finitely generated R -module M , where $r < \dim_R M$. We will show that if (y_1, \dots, y_r) is part of a reducing system of parameters of M with $(y_1, \dots, y_r)M = (x_1, \dots, x_r)M$ then (x_1, \dots, x_r) is already reducing. Moreover, there is such a part of a reducing system of parameters of M iff for all primes $P \in \text{Supp } M \cap V_R(x_1, \dots, x_r)$ with $\dim_R R/P = \dim_R M - r$ the localization M_P of M at P is an r -dimensional Cohen–Macaulay module over R_P .

Furthermore, we will show that M is a Cohen–Macaulay module iff y_d is a non zero divisor on $M/(y_1, \dots, y_{d-1})M$, where (y_1, \dots, y_d) is a reducing system of parameters of M ($d := \dim_R M$).

Keywords. Systems of parameters; Cohen–Macaulay modules.

1. Preliminaries

In what follows, let R be a local ring with maximal ideal \mathfrak{m} and let M be a non zero finitely generated R -module of dimension d . Instead of \dim_R , depth_R , Ass_R , Supp_R , ... we will write \dim , depth , Ass , Supp , ... for short.

We note that $\text{Supp } M/XM = \text{Supp } M \cap V(X)$, where X is a subset of R and that for a prime ideal P of R we have $P \in \text{Ass } M$ iff $PR_P \in \text{Ass } M_P$. Moreover we define $\text{Assh } M := \{P \in \text{Ass } M \mid \dim R/P = d\}$.

For undefined terminology we refer to the standard literature (e.g. [E]).

DEFINITION 1.

A system of parameters (x_1, \dots, x_d) of M is called *reducing*, if for all $i = 1, \dots, d-1$ we have

$$x_i \notin P \text{ for all } P \in \text{Ass } M/(x_1, \dots, x_{i-1})M \text{ with } \dim R/P = d-i.$$

Remark 2. Auslander and Buchsbaum defined in [AB] a system of parameters (x_1, \dots, x_d) of R to be a reducing system of parameters of M if

$$\begin{aligned} e_M(x_1, \dots, x_d) &= \text{length}(M/(x_1, \dots, x_d)M) \\ &\quad - \text{length}((x_1, \dots, x_{d-1})M : x_d/(x_1, \dots, x_{d-1})M). \end{aligned}$$

This definition is equivalent to the definition given above if we pass from R to $\bar{R} := R/\text{Ann}_R M$ and consider M as a \bar{R} -module in Definition 1 and use Corollary 4.8 in [AB]. Therefore it is clear that all definitions and results on reducing systems of parameters remain true in this more general context.

Remark 3. For every system of parameters (x_1, \dots, x_d) of M there is a reducing system of parameters (y_1, \dots, y_d) of M such that $(y_1, \dots, y_d)R = (x_1, \dots, x_d)R$, in particular, $(y_1, \dots, y_d)M = (x_1, \dots, x_d)M$ (see Proposition 4.9 in [AB]).

DEFINITION 4.

A sequence x_1, \dots, x_r of elements of \mathfrak{m} is called *part of a (reducing) system of parameters* of M , if there are elements $x_{r+1}, \dots, x_d \in \mathfrak{m}$ such that $(x_1, \dots, x_r, x_{r+1}, \dots, x_d)$ is a (reducing) system of parameters of M .

Remark 5.

- (1) A sequence (x_1, \dots, x_r) of elements of \mathfrak{m} with $r < d$ is part of a system of parameters of M iff $\dim M/(x_1, \dots, x_r)M = d - r$.
- (2) A sequence (x_1, \dots, x_r) of elements of \mathfrak{m} with $r < d$ is part of a reducing system of parameters of M iff for all $i = 1, \dots, r$ we have $x_i \notin P$ for all $P \in \text{Ass} M/(x_1, \dots, x_{i-1})M$ with $\dim R/P \geq d - i$.
- (3) Every regular sequence on M is part of a reducing system of parameters of M .

Remark 6.

- (1) We note that the following conditions are equivalent:
 - (i) M is a Cohen–Macaulay module, i.e. $\text{depth} M = d$.
 - (ii) Every system of parameters of M is a regular sequence on M .
 - (iii) There exists a system of parameters of M which is a regular sequence on M .
- (2) Assume that M is a Cohen–Macaulay module. If (x_1, \dots, x_r) is part of a system of parameters of M then $M/(x_1, \dots, x_r)M$ is unmixed, more precisely, $\dim R/P = d - r$ for all $P \in \text{Ass} M/(x_1, \dots, x_r)M$. Therefore for a sequence (x_1, \dots, x_r) of elements of \mathfrak{m} the following conditions are equivalent:
 - (i) (x_1, \dots, x_r) is a regular sequence on M .
 - (ii) (x_1, \dots, x_r) is part of a reducing system of parameters of M .
 - (iii) (x_1, \dots, x_r) is part of a system of parameters of M .

Let $x_1, \dots, x_r \in \mathfrak{m}$. If (x_1, \dots, x_r) is a regular sequence on M then (x_1, \dots, x_r) is a regular sequence on M_P as well for all primes $P \in \text{Supp} M \cap V(x_1, \dots, x_r)$.

Lemma 7. Let (x_1, \dots, x_r) be part of a (reducing) system of parameters of M . Then (x_1, \dots, x_r) is part of a (reducing) system of parameters of M_P for all primes $P \in \text{Supp} M \cap V(x_1, \dots, x_r)$ with $\dim R/P + \dim M_P = d$.

Proof. Let $P \in \text{Supp} M \cap V(x_1, \dots, x_r)$ with $\dim R/P + \dim M_P = d$. An easy induction argument (induction on r) shows that we can restrict ourselves to the case $r = 1$ (and $\dim M_P \geq 2$).

Let $\mathfrak{q} \in \text{Ass} M_P$ with $\dim R_P/\mathfrak{q} = \dim M_P$ ($\dim R_P/\mathfrak{q} \geq \dim M_P - 1$). Then $\mathfrak{q} = QR_P$ with $Q \in \text{Ass} M$, $Q \subseteq P$, and we obtain

$$\begin{aligned} \dim R/Q &\geq \dim R/P + \dim(R/Q)_P = \dim R/P + \dim R_P/\mathfrak{q} \\ &= \dim R/P + \dim M_P = d \\ &(\geq \dim R/P + \dim M_P - 1 = d - 1). \end{aligned}$$

Therefore $x_1 \notin Q$ by our assumption. But then $x_1 \notin \mathfrak{q}$, i.e. (x_1) is part of a (reducing) system of parameters of M_P . \square

Lemma 8. *If $x \in R$ is a zero divisor on M , then $P \in \text{Ass} M/xM$ for all minimal primes $P \in \text{Ass} M \cap V(x)$.*

Proof. Let $P \in \text{Ass} M \cap V(x)$ be minimal. Since $P \in \text{Ass} M/xM$ iff $PR_P \in \text{Ass} M_P/xM_P$ we may assume by localizing at P that $P = \mathfrak{m}$. Then $x \notin Q$ for all $Q \in \text{Ass} M \setminus \{\mathfrak{m}\}$. Since R is noetherian there is an $i \in \mathbb{N}^+$ such that $0 :_M x^i = 0 :_M x^j$ for all $j \geq i$. Let $U := 0 :_M x^i$. Then $U \neq 0$ (otherwise x would be a non zero divisor on M , contradicting our assumption).

Let $Q \in \text{Supp} M \setminus \{\mathfrak{m}\}$. Since $\text{Ass} M_Q = \{Q'R_Q | Q' \in \text{Ass} M, Q' \subseteq Q\}$, we have $x \notin \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass} M_Q$. Therefore $U_Q = 0 :_{M_Q} x^i = 0$ for all $Q \in \text{Supp} M \setminus \{\mathfrak{m}\}$, i.e. $\text{Supp} U = \{\mathfrak{m}\}$. Moreover,

$$U :_M x = 0 :_M x^{i+1} = 0 :_M x^i = U.$$

Let $\varphi: U \rightarrow M/xM$ be the inclusion $U \subseteq M$ followed by the canonical epimorphism MM/xM . Since

$$\ker \varphi = U \cap xM = x(U :_M x) = xU,$$

φ induces a monomorphism $U/xU \rightarrow M/xM$. Now $U/xU \neq 0$ by Nakayama's lemma. Therefore $\emptyset \neq \text{Ass} U/xU \subseteq \text{Ass} U = \{\mathfrak{m}\}$, that means $\text{Ass} U/xU = \{\mathfrak{m}\}$. This gives us the existence of a monomorphism $R/\mathfrak{m} \rightarrow U/xU \rightarrow M/xM$. Thus $\mathfrak{m} \in \text{Ass} M/xM$. \square

Lemma 9. *Let $Q \in \text{Supp} M$ and assume that there is an $x \in \mathfrak{m}$ with $x \notin Q$. Then there is a $P \in \text{Supp} M$ such that $x \in P$, $Q \subset P$ and $\dim R/P = \dim R/Q - 1$.*

Proof. Since (x) is part of a system of parameters of R/Q , there is a $P \in \text{Supp}(R/Q)/x(R/Q) = \text{Supp} R/(Q + xR) = V(Q + xR) \subset V(Q)$ with $\dim R/P = \dim R/(Q + xR) = \dim R/Q - 1$. Since $0 \neq M_Q \cong (M_P)_{QR_P}$ we have $M_P \neq 0$, i.e. $P \in \text{Supp} M$. \square

COROLLARY 10.

Let $Q \in \text{Supp} M$ and let $x_1, \dots, x_r \in \mathfrak{m}$. Then there is a $P \in \text{Supp} M \cap V(x_1, \dots, x_r)$ such that $Q \subseteq P$ and $\dim R/P \geq \dim R/Q - r$.

The proof follows immediately from Lemma 9 by induction on r .

2. Main results

Theorem 11. *Let (y_1, \dots, y_d) be a reducing system of parameters of M . M is a Cohen–Macaulay module iff y_d is a non zero divisor on $M/(y_1, \dots, y_{d-1})M$.*

Proof. The implication ‘ \Rightarrow ’ is clear, since every system of parameters in a Cohen–Macaulay module is a regular sequence (see Remark 6(1)).

We will prove the opposite implication by induction on d , where the case $d = 1$ is clear. Let $d \geq 2$ and assume that the statement is true for modules with a dimension strictly less than d .

Assume that y_d is a non zero divisor on $M/(y_1, \dots, y_{d-1})M$. By our induction hypothesis, M/y_1M is a Cohen–Macaulay module and therefore it remains to show that y_1 is a non zero divisor on M . Suppose this is not the case. Let P be minimal in $\text{Ass}M \cap V(y_1)$. Since (y_1) is part of a reducing system of parameters of M , we have $\dim R/P \leq d - 2$. By Lemma 8, $P \in \text{Ass}M/y_1M$ and therefore $\dim R/P = \dim M/y_1M = d - 1$ (see Remark 6(2)), a contradiction. \square

Lemma 12. *Let (x) be part of a system of parameters of M . If $d \geq 2$, the following conditions are equivalent:*

- (i) (x) is part of a reducing system of parameters of M .
- (ii) M_P is a one-dimensional Cohen–Macaulay module over R_P for all $P \in \text{Supp}M \cap V(x)$ satisfying $\dim R/P = d - 1$.
- (iii) There is a $y \in \mathfrak{m}$ such that (y) is part of a reducing system of parameters of M and $yM = xM$.
- (iv) There is a $y \in \mathfrak{m}$ such that (y) is part of a reducing system of parameters of M and $\text{Supp}M \cap V(x) \subseteq V(y)$.

Proof. The implications (i) \Rightarrow (iii) and (iii) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (ii): Let (y) be part of a reducing system of parameters of M with $\text{Supp}M \cap V(x) \subseteq V(y)$ and let $P \in \text{Supp}M \cap V(x)$ with $\dim R/P = d - 1$. Then $y \in P$. Now $y \notin Q$ for all $Q \in \text{Ass}M$ with $\dim R/Q \geq d - 1$ by our assumption. Thus $P \notin \text{Ass}M$ and therefore $PR_P \notin \text{Ass}M_P (\neq \emptyset)$ from which

$$0 < \text{depth}M_P \leq \dim M_P \leq \dim M - \dim R/P = 1,$$

i.e. $\text{depth}M_P = \dim M_P = 1$.

(ii) \Rightarrow (i): Let $P \in \text{Ass}M$ with $\dim R/P \geq d - 1$. If $\dim R/P = d$, then $x \notin P$ since (x) is part of a system of parameters of M . Let $\dim R/P = d - 1$. If $x \in P$, then M_P is a Cohen–Macaulay module over R_P with $\dim M_P = 1$. Therefore $PR_P \notin \text{Ass}M_P$ contradicting $P \in \text{Ass}M$.

Thus $x \notin P$ for all $P \in \text{Ass}M$ with $\dim R/P \geq d - 1$, i.e. (x) is part of a reducing system of parameters of M by Remark 5(2). \square

Remark 13. Let $x_1, \dots, x_r, y_1, \dots, y_r$ be elements of \mathfrak{m} with

$$\text{Supp}M \cap V(x_1, \dots, x_r) \subseteq V(y_1, \dots, y_r)$$

(which is equivalent to $\text{Supp}M/(x_1, \dots, x_r)M \subseteq \text{Supp}M/(y_1, \dots, y_r)M$).

- (a) If (y_1, \dots, y_r) is part of a system of parameters of M then the same is true for (x_1, \dots, x_r) . This follows immediately from Remark 5(1).
- (b) If (y_1, \dots, y_r) is a regular sequence on M then the same is true for (x_1, \dots, x_r) . This follows from Corollary 2 of [PSS].

The equivalence (i) \Leftrightarrow (iv) of our next theorem shows that a similar statement holds for parts of reducing systems of parameters of M , provided $r < d$. (For $r = d$ this is not true in general, see Remark 3.)

Theorem 14. *Let (x_1, \dots, x_r) be part of a system of parameters of M , where $0 \leq r < d$. Then the following conditions are equivalent:*

- (i) (x_1, \dots, x_r) is part of a reducing system of parameters of M .
- (ii) M_P is an r -dimensional Cohen–Macaulay module over R_P for all $P \in \text{Supp } M \cap V(x_1, \dots, x_r)$ satisfying $\dim R/P = \dim M - r$.
- (iii) There is a part (y_1, \dots, y_r) of a reducing system of parameters of M such that $(y_1, \dots, y_r)M = (x_1, \dots, x_r)M$.
- (iv) There is a part (y_1, \dots, y_r) of a reducing system of parameters of M such that $\text{Supp } M \cap V(x_1, \dots, x_r) \subseteq V(y_1, \dots, y_r)$.

Proof. We use induction on r . For $r = 0$, there is nothing to show and for $r = 1$ the statement follows from Lemma 12. So let $r \geq 2$.

The implications (i) \Rightarrow (iii) and (iii) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (ii): Let $\bar{M} := M/y_1M$. Take $P \in \text{Supp } M \cap V(x_1, \dots, x_r) \subseteq \text{Supp } M \cap V(y_1, \dots, y_r)$ with $\dim R/P = d - r$. Since $\dim \bar{M} = d - 1$, $\bar{M}_P \cong M_P/y_1M_P$ is an $(r - 1)$ -dimensional Cohen–Macaulay module (over R_P) by the induction hypothesis ((i) \Rightarrow (ii)). Therefore it is sufficient to show that y_1 is a non zero divisor on M_P .

Suppose this is not the case. Then by Lemma 8 there is a $\mathfrak{q} \in \text{Ass } M_P$ with $\mathfrak{q} \in \text{Ass } \bar{M}_P$. Therefore $\dim R_P/\mathfrak{q} = r - 1$ by Remark 6(2). Now $\mathfrak{q} = QR_P$ with $Q \in \text{Supp } \bar{M} = \text{Supp } M \cap V(y_1)$ and $Q \subseteq P$. Then $Q \in \text{Ass } M$ and we have

$$\dim R/Q \geq \dim R/P + \dim(R/Q)_P = \dim R/P + \dim R_P/\mathfrak{q} = d - 1.$$

Therefore $y_1 \notin Q$ (since (y_1, \dots, y_r) is part of a reducing system of parameters of M), a contradiction.

(ii) \Rightarrow (i): Let $\bar{M} := M/x_1M$ and take $P \in \text{Supp } \bar{M} \cap V(x_2, \dots, x_r) = \text{Supp } M \cap V(x_1, \dots, x_r)$ with $\dim R/P = \dim \bar{M} - (r - 1) = \dim M - r$. Then \bar{M}_P is an r -dimensional Cohen–Macaulay module (over R_P) by our assumption and (x_1, \dots, x_r) is a system of parameters of M_P and hence a regular sequence on M_P by Remark 6(1). But then $\bar{M}_P \cong M_P/x_1M_P$ is an $(r - 1)$ -dimensional Cohen–Macaulay module (over R_P).

By the induction hypothesis (x_2, \dots, x_r) is part of a reducing system of parameters of \bar{M} and therefore it remains to show that $x_1 \notin Q$ for all $Q \in \text{Ass } M$ with $\dim R/Q = d - 1$.

Suppose this is not the case. Choose $Q \in \text{Ass } M$ with $\dim R/Q = d - 1$ and $x_1 \in Q$. By Corollary 10 there is a prime $P \in \text{Supp } M \cap V(x_2, \dots, x_r)$ such that $Q \subseteq P$ and $\dim R/P \geq d - 1 - (r - 1) = d - r$. But then $P \in \text{Supp } M \cap V(x_1, \dots, x_r)$ (since $x_1 \in Q \subseteq P$) and therefore $\dim R/P \leq d - r$, i.e. $\dim R/P = d - r$. By our assumption, M_P is an r -dimensional Cohen–Macaulay module. Since $QR_P \in \text{Ass } M_P$ we therefore have

$$\begin{aligned} r = \dim M_P &= \dim R_P/QR_P = \dim(R/Q)_P \\ &\leq \dim R/Q - \dim R/P = d - 1 - (d - r) \\ &= r - 1 \end{aligned}$$

(see Remark 6(2)), a contradiction. \square

COROLLARY 15.

Let (x_1, \dots, x_r) be part of a reducing system of parameters of M . If $r < d$, then $(x_{\pi(1)}, \dots, x_{\pi(r)})$ is part of a reducing system of parameters of M for any permutation π of $\{1, \dots, r\}$.

We note that the statement of this corollary is not true in general if $r = d$, see the following Example 16.

Example 16. Let $R := K[[X, Y, Z]]$, where K is a field and X, Y, Z are indeterminates. For

$$M := R/(XY, XZ)R \quad \text{and} \quad x_1 := Y, x_2 := X + Y + Z,$$

(x_1, x_2) is a system of parameters of M , but not a reducing system of parameters. (x_2, x_1) is a reducing system of parameters of M (not a regular sequence of M).

Finally we define the following.

DEFINITION 17.

$$\mathcal{CM}(M) := \{P \in \text{Supp} M \mid \dim R/P + \dim M_P = d \text{ and } M_P \text{ is a Cohen–Macaulay module over } R_P\}$$

(the *strong Cohen–Macaulay locus* of $\text{Supp} M$) and for $0 \leq r \leq d$

$$\mathcal{CM}_r(M) := \{P \in \mathcal{CM}(M) \mid \dim M_P = r\}.$$

Remark 18.

(1) We have

- (i) $\mathcal{CM}_0(M) = \text{Assh} M$ and, if $d \geq 1$,
- (ii) $\mathcal{CM}_1(M) = \{P \in \text{Supp} M \mid \dim R/P = d - 1\} \setminus \text{Ass} M$,
- (iii) $\mathcal{CM}(M) = \bigcup_{r=0}^d \mathcal{CM}_r(M)$.

(2) The following conditions are equivalent

- (i) M is a Cohen–Macaulay module,
- (ii) $\mathcal{CM}(M) = \text{Supp} M$,
- (iii) $\mathfrak{m} \in \mathcal{CM}(M)$.

(3) If $\text{Supp} M$ is equidimensional and catenarian then $\mathcal{CM}(M)$ coincides with the ordinary Cohen–Macaulay locus of $\text{Supp} M$. This is the case, for example, when $\dim M \leq 1$ or when R is an epimorphic image of a local Cohen–Macaulay ring and M is equidimensional.

PROPOSITION 19.

For $r \in \mathbb{N}$, $r < d$, we have

$$\mathcal{CM}_r(M) = \{P \mid P \in \text{Ass} M/(x_1, \dots, x_r)M, \dim R/P = d - r, (x_1, \dots, x_r) \text{ part of a reducing system of parameters of } M\}.$$

Proof. By Theorem 14 we have ‘ \supseteq ’ and equality holds (trivially) for $r = 0$. Therefore it remains to verify the validity of the inclusion ‘ \subseteq ’ for $r \geq 1$.

Let $P \in \mathcal{C}\mathcal{M}_r(M)$. Since M_P is a Cohen–Macaulay module with $\dim M_P = r \geq 1$, we have $PR_P \notin \text{Ass } M_P$ and hence $P \notin \text{Ass } M$. Moreover, $\dim R/P = d - \dim M_P = d - r$.

Let $Q \in \text{Ass } M$ with $\dim R/Q \geq d - 1$. Then $P \not\subseteq Q$ since $P = Q$ is impossible ($P \notin \text{Ass } M$) and $P \subset Q$ would imply $\dim R/P = d$ contradicting again ‘ $P \notin \text{Ass } M$ ’. Therefore we can find an $x_1 \in P$ with $x_1 \notin Q$ for all $Q \in \text{Ass } M$ with $\dim R/Q \geq d - 1$. By construction, (x_1) is part of a reducing system of parameters of M and a regular sequence on M_P by Lemma 7 and Remark 6(2).

If $r > 1$ we continue this procedure by passing to M/x_1M and we can construct elements $x_1, \dots, x_r \in P$ inductively on r such that (x_1, \dots, x_r) forms a part of a reducing system of parameters of M and a regular sequence on M_P . Let $\tilde{M} := M/(x_1, \dots, x_r)M$. Since $\dim \tilde{M}_P = \dim M_P/(x_1, \dots, x_r)M_P = \dim M_P - r = 0$, P is minimal in $\text{Supp } \tilde{M}$ and therefore $P \in \text{Ass } \tilde{M}$. \square

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